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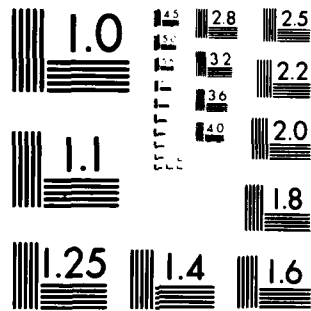
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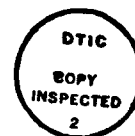
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20. Abstract (continued)

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OPTIMUM SAMPLING TIMES FOR SPECTRAL ESTIMATION

by

Lonnie C. Ludeman

ABSTRACT

The problem of optimum sampling strategies for spectral estimation of Fourier-type signals in the case of finite discrete-time observations was investigated. In particular, it was shown that minimum variance unbiased estimates of amplitudes of sine and cosine terms of Fourier signals embedded in additive zero-mean white noise can be determined by sampling at the generalized Chebyshev times. The solution obtained, by putting the problem in an optimum linear regression framework, is that the generalized Chebyshev times are the zeros of the derivative of the highest frequency cosine wave. If the number of samples exceeds the intrinsic dimensionality, repeated independent sampling at those points not only provides the best approximation to the Fourier signal in a minimum variance sense but also the linear minimum variance unbiased estimate of the coefficients.

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I. INTRODUCTION

In obtaining spectral information from noisy discrete-time measurements of Fourier signals, a basic question is whether performance can be improved by carefully selecting the sampling times. Preliminary results shown by the author [8,9] in pursuit of the answer to this question helped formulate the main objectives of this research effort. The objectives were:

- (1) Find the optimum sampling strategy for the case of minimum variance unbiased estimation of (a) the amplitudes of several sine and cosine terms of harmonically related frequencies in white noise; (b) the amplitudes of Fourier-type nonharmonically related signals in additive white noise; and (c) spectral content of a low pass process in additive white noise.
- (2) Repeat (1) for special types of nonwhite additive noise.
- (3) Find the improvement obtained with nonuniform oversampling for the cases given in (1) a, b, and (2) a, b.
- (4) Show if possible under what conditions the uniform sampling value estimates, determined from nonuniform sample times, provide a sufficient statistic for the spectral estimation problem.

During the grant period progress was made in areas specified in Objectives 1, 2 and 3, while the sufficient statistic problem given in 4 has not been resolved. In the sections to follow, optimum sampling strategies will be shown for both Fourier signals and Fourier-type signals in additive white noise ((1) a, b) from two different approaches.

The first approach, the main body of the report, places the problem in a linear regression framework and uses the elegant and brilliant work of Kiefer and Wolfowitz [2] to arrive at the solution that the optimum sample times are specified by a generalized Chebyshev set.

The second approach--more of a brute force method--is that of minimizing the trace of the error covariance matrix as presented in Appendix A. Optimum sampling points were obtained by applying standard search and steepest descent optimization algorithms to the multidimensional problem. This approach was complicated by the dimensionality, the intrinsic dimensionality, and the preponderance of local rather than global minimums.

The nonwhite additive noise problem was approached again both by computer simulation (Appendix A) and from the standpoint of changing the colored noise to white noise via a bleaching filter and searching for the Chebyshev points of the bleached basis. Computer results showed equal spacing for the three sample case of a signal composed of a sum of a sine and cosine wave for colored noise with exponential autocorrelation function.

Progress on Objective (3) has been minimal, but because of results of Kiefer and Wolfowitz [2] it is believed that for large numbers of samples, much more than the intrinsic dimensionality of the problem, uniform sampling strategy requires roughly $8/\pi$ times the number of samples in the optimum strategy to yield identical performance. The optimal sampling strategy may in some sense not be physically meaningful in that it requires taking more than one sample at several points or times; however, because of periodicity of the assumed signals this could mean simply using samples in a later period.

II. PROBLEM FORMULATION

The common problem of spectral estimation involves the estimation of the frequency content of a given signal. Normally the signal is not available over all time and has been contaminated with additive noise. If we assume further that the processing is to be done digitally, we have available only measured values at some finite set of times. The problem we wish to address is that of optimum sampling strategies for spectral estimation of Fourier-type signals. The K^{th} order Fourier signal (trigonometric polynomial) is given by

$$x_K(t) = \frac{A_0}{2} + \sum_{k=1}^K A_k \cos k\pi t + \sum_{k=1}^K B_k \sin k\pi t \quad (1)$$

The A_0, A_1, \dots, A_K and B_1, B_2, \dots, B_K represent unknown nonrandom Fourier coefficients. It is further assumed that we have available $z(t)$ over an interval -1 to 1 , given by

$$z(t) = x_K(t) + N(t) \quad -1 \leq t \leq 1 \quad (2)$$

where $N(t)$ is a random noise process characterized by its autocorrelation function. Let T_M be an arbitrary set of M time values given by

$$T_M = \{t_1, t_2, \dots, t_M\}$$

Observing $z(t)$ at the times of T_M where M is greater than or equal to $2K + 1$ gives the set of measurements

$$z(t_i) = x_K(t_i) + N(t_i) \quad i = 1, 2, \dots, M \geq 2K+1 \quad (3)$$

We would like to select T_M in such a way that optimal estimates of $A_0, A_1, \dots, A_K, B_1, B_2, \dots, B_K$ or $x_K(t)$ are obtained where optimality is with respect to a certain performance criterion and a given M . Two prominent criteria are 1) minimax variant which is to minimize the maximum variance of the estimate, $\hat{x}_K(t)$ over all $t \in [-1, 1]$ using Lagrangian interpolation, and 2) minimum trace of the error covariance matrix, that is, minimizing the sum of the variances of the estimates of the coefficients. The first criterion will be used in the body of this report, while the results of the second are given in the appendix for completeness.

III. OPTIMUM POLYNOMIAL REGRESSION

The optimum spacing problem for finite polynomial regression has been investigated by De La Garza [4,5], Hoel and Levine [6], and Kiefer and Wolfowitz [3]. The framework for their development is as follows: Let $P(x)$ be a polynomial given by

$$P(x) = \alpha_1 + \alpha_2 x + \dots + \alpha_{m+1} x^m \quad m \geq 1 \quad (4)$$

and let $y(x_i)$ represent observed variates given by

$$y(x_i) = P(x_i) + \delta_i \quad i = 1, 2, \dots, N, \quad N \geq (m+1) \quad (5)$$

where δ_i are uncorrelated random errors with known variances and zero means.

It was shown that given a selection of $N \geq (m+1)$ distinct points x_i , with minimum $x_i = -1$, maximum $x_i = 1$, it is always possible to respace at $k+1$ distinct location with repeated values of x_i to yield identical performance. Therefore, for optimum selection you need only find those $(m+1)$ points and the number of times each was repeated. The points of observation and the number of times n_i for minimax variance as given by Kiefer [2] are

$$x_0 = -1, \quad x_j = -\cos j \frac{\pi}{m}, \quad j = 1, 2, \dots, m-1, \quad x_m = 1, \text{ with} \quad (6)$$

$$n_0 = \frac{N}{2m}, \quad n_i = \frac{N}{m}, \quad i = 1, \dots, m-1, \quad n_m = \frac{N}{2m} \quad (7)$$

This result will now be applied to the special case where $x_K(t)$ contains only harmonic cosine waves.

Example 1. Let $x_{KC}(t)$ represent a series of harmonic cosine waves given by

$$x_{KC}(t) = A_0 + \sum_{k=1}^m A_k \cos k\pi t \quad (8)$$

It is well known that each of the cosine waves $\cos k\pi t$ can be written in terms of powers of $\cos \pi t$, the fundamental, by repeated use of the recursive relation

$$\cos k\pi t = 2\cos(k-1)\pi t \cos \pi t - \cos(k-2)\pi t$$

That is, $x_{KC}(t)$ can be written as a K^{th} order polynomial in $\cos \pi t$ given by

$$x_{Kc}(t) = \alpha_0 + \alpha_1 \cos \pi t + \alpha_2 (\cos \pi t)^2 + \dots + \alpha_K (\cos \pi t)^m \quad (9)$$

By letting $\cos t = x$, we have the $-1 \leq x \leq 1$ and the polynomial condition necessary to find the optimal x and thus the corresponding optimal t from the polynomial regression results given by equation (6). This yields

$$\cos \pi t = -1 \quad \cos \pi t = -\cos j \frac{\pi}{m} \quad j = 1, \dots, m-1 \quad \cos \pi t = 1 \quad (10)$$

Therefore we see the solution for the t_j , after reordering, is as follows:

$$t_j = \frac{j}{m} \quad j = 0, 1, \dots, m \quad \text{on } [0, 1] \quad (11)$$

Computer results using the formulation given in Appendix A for the case of a cosine wave and its first harmonic showed that the optimum points on the interval $[0, 1)$ were $t = 0.0$ and $t = 0.5$ with these same times repeated when the number of points taken was greater than 2. If, however, harmonics were skipped--for example, using only the first and third--these points were no longer optimum. If harmonics are skipped or not present, the intrinsic dimensionality of the problem has been reduced and the resulting cosine functions used for the basis in the generalized regression are no longer Chebyshev.

Computer results showed also that for two cosine waves not harmonically related, the optimum times were equally spaced if the waves were Chebyshev.

IV. GENERALIZED OPTIMUM REGRESSION

In Kiefer and Wolfowitz [2], the optimum selection of regressor variables was extended to cover other nonpolynomial regression problems involving Chebyshev systems. In this section we will summarize the important results of their extension and later apply them to the problems of spectral estimation.

The regression problem considered assumes the observed function $z(t)$ is given by

$$z(t) = \sum_{i=0}^m \theta_i f_i(t) + N(t) \quad -1 \leq t \leq 1 \quad (12)$$

where the $f_i(t)$ are continuous and linearly independent real-valued functions on $[-1,1]$. We would like to select t_i , $i=0,1,\dots,n \geq m+1$ such that the maximum variance on $z(t)$ over T is minimized where $T=\{t: -1 \leq t \leq 1\}$.

The functions $f_0(t), f_1(t), \dots, f_m(t)$ are called a Chebyshev system

on a set U if every linear combination $\sum_{k=0}^m c_k f_k(t)$, with not all of the real

constants c_i zero, has k or fewer zeros on U . Several other equivalent specifications are given by K and W [3]. We shall denote $B(c^*)$ to be the

set of $m+1$ points where $|f_m(t) - \sum_{k=0}^{m-1} c_k^* f_k(t)|$ attains its maximum on $[-1,1]$

and $c^* = (c_0^*, c_1^*, \dots, c_{m-1}^*)$ as the constants $c = (c_0, c_1, \dots, c_{m-1})$ such

that $\max_{-1 \leq t \leq 1} |f_m(t) - \sum_{j=0}^{m-1} c_j f_j(t)|$ is minimized.

$B(c^*)$ consists of the $m+1$ points given by

$$t_0^* = -1, \quad t_0^* < t_1^* < \dots < t_{m-1}^*, \quad t_m^* = 1.$$

Kiefer and Wolfowitz have shown for the given assumptions that the c_j^* must satisfy

$$\int_{-1}^1 [f_m(t) - \sum_{j=0}^{m-1} c_j^* f_j(t)] f_i(t) \xi^*(dx) = 0 \quad 0 \leq i \leq m \quad (13)$$

For the case where the probability measure ξ is uniform, we have

$$\int_{-1}^1 [f_m(t) - \sum_{j=0}^{m-1} c_j^* f_j(t)] f_i(t) dt = 0 \quad 0 \leq i < m \quad (14)$$

If we further assume the $\{f_j(t), j=0,1,\dots,m\}$ to be orthogonal on $[-1,1]$, i.e.,

$$\int_{-1}^1 f_j(t) f_k(t) dt = w_j \delta_{jk} \quad \text{for } j,k = 0,1,\dots, m \quad (15)$$

where $w_j > 0$

then the set of equations in (14) reduce to

$$c_i^* w_i = 0 \quad 0 \leq i < m$$

Since w_j are nonzero it implies that $c_i^* = 0$ for $i=0,1,\dots,m-1$. With this result $B(c^*)$ is seen to be from [3] to be the place where $|f_m(t)|$ attains its maximum values on $(-1,1)$.

Application to Spectral Estimation White Noise

The above result is now applied to the case where $f_i(t)$ are the Fourier Basis functions on $-1 \leq t \leq 1$ given by

$$\begin{aligned} f_0(t) &= 1 \\ f_k(t) &= \sin \pi t \\ f_2(t) &= \cos \pi t \\ f_3(t) &= \sin 2\pi t \\ f_4(t) &= \cos 2\pi t \\ &\vdots \\ f_{m-1}(t) &= \sin\left(\frac{m}{2}\right)\pi t \\ f_m(t) &= \cos\left(\frac{m}{2}\right)\pi t \end{aligned}$$

We wish to select the times t such that the maximum variance is minimized throughout -1 to 1 . The result states that the optimum times will be the points at which $|\cos \frac{m}{2}\pi t|$ reaches maximum, that is, where t equals -1 and 1 and $\sin \frac{m}{2}\pi t$ equals zero. Therefore the times are given by

$$\frac{m}{2}\pi t = \begin{cases} j\pi \\ 0 \\ -j\pi \end{cases} \quad \text{which implies } t = \begin{cases} 2j/m & j = 0, 1, \dots, m/2 \\ 0 & \\ -2j/m & j = 0, 1, \dots, m/2 \end{cases}$$

These points are illustrated in Figure 1 below.

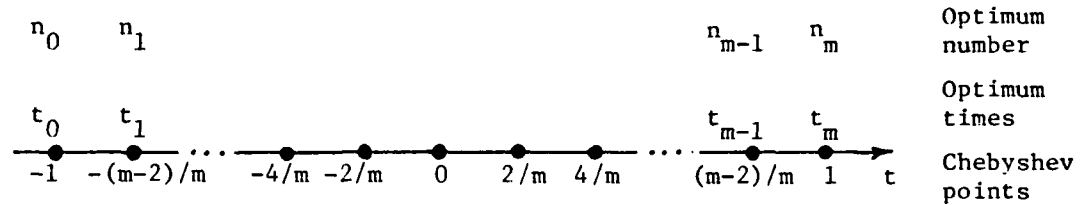


Figure 1. Optimum Sample Times for Estimation of Fourier Signals: $m+1$ Chebyshev Points with N Total Samples.

For N samples, where N is greater than $m+1$, the above Chebyshev points will have weights $\xi_0, \xi_1, \dots, \xi_m$ given by

$$\xi_0 = \frac{1}{2m}, \quad \xi_i = \frac{1}{m}, \quad i = 1, 2, \dots, m-1, \quad \xi_m = \frac{1}{2m}$$

If N is the total number of points, then n_j , the number of samples to be taken at t_j , is $N \cdot \xi_j$. Furthermore, if we let $N = 2m \cdot r$ where r is an integer, we have

$$n_0 = r, \quad n_i = 2r, \quad i = 1, 2, \dots, m-1, \quad n_m = r$$

The variance at each point and thus the minimum maximum variance becomes σ^2/n_j where σ^2 is variance of each observation and n_j is the number of measurements taken at each point in the interior of -1 and 1 . When N is a multiple of $2m$ as above, we get

$$\text{minimax variance} = \frac{m\sigma^2}{N} = \frac{\sigma^2}{2r}$$

Kiefer and Wolfowitz[2] defined a sampling efficiency and showed for polynomial regression that the evenly spaced samples strategy requires roughly $8/\pi$ or about $5/2$ times that of Chebyshev points strategy for large N .

Application to Spectral Estimation: Colored Noise

If the $N(t)$ given in equation (12) is not white but its autocorrelation function is known, one approach to finding the optimum sampling times is to change the problem to white noise by "bleaching." We assume that $N(t)$ can be obtained by passing a white noise process $w(t)$ through a time invariant linear filter with system function $H(s)$. The whitening filter would be the reciprocal of $H(s)$ and would operate on $z(t)$ to give a new measurement process $z^b(t)$ shown in Figure 2.

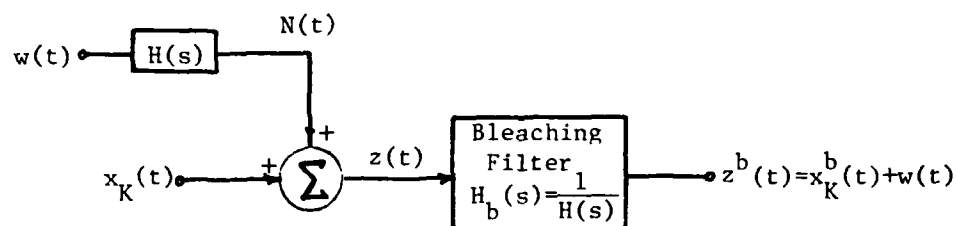


Figure 2. Conversion to White Noise Problem by Bleaching.

The $z^b(t)$ is composed of $w(t)$, a white noise process, and a bleached version $x_K^b(t)$ of $x_K(t)$. If we allow the filtering to reach a steady state, $z_{ss}^b(t)$ can be written as

$$\begin{aligned}
 z_{ss}^b(t) = & \frac{A_o}{2} \cdot H_b(0) + \sum_{k=1}^K |H_b(jk\omega_o)| \cos\left(k\omega_o t + \tan^{-1} \frac{H_{bI}(jk\omega_o)}{H_{bR}(jk\omega_o)}\right) \\
 & + \sum_{k=1}^K |H_b(jk\omega_o)| \sin\left(k\omega_o t + \tan^{-1} \frac{H_{bI}(jk\omega_o)}{H_{bR}(jk\omega_o)}\right) + w(t) \quad (16)
 \end{aligned}$$

+ w(t) where $\omega_o = \pi$

We now have the same framework as before for optimum linear regression. It is easy to show that the translated sine and cosine functions are linearly independent and Chebyshev, thus specifying the optimum sampling times for the steady state bleached $z(t)$. The performance, however, of the estimates will be modified by the fact that the bleaching process changes the variance of the $N(t)$ by $|1/H(j0)|^2$. Computer results for the case of colored noise with autocorrelation function $R_N(\tau) = e^{-\alpha\tau}$ and a first order trigonometric

polynomial showed that equally spaced samples were optimum. Results for higher order trigonometric polynomials and over sampling were not obtained.

V. CONCLUSION

It has been shown that optimum sampling strategies exist for estimating Fourier signals using a finite number of discrete time observations. By framing the problem as a generalized regression with a Chebyshev basis, it has been shown that the generalized Chebyshev times are the zeros of the highest sine wave considered. Therefore, the samples are equally spaced with the number of unique samples being determined by the dimensionality of the trigonometric polynomials. For the case of a Fourier series of order m , with no missing harmonics, the sample points on -1 to 1 are $\pm \frac{2j}{m}$, $j = 0, \dots, m/2$. If harmonics were missing, the dimensionality of the problem has been reduced and a fine structure appears. When a Fourier-type signal was used rather than a harmonic structure results again indicate the uniform spaced property and repeated sampling are optimum provided the signal components are Chebyshev. If the observations are obtained in colored noise, with a known exponential autocorrelation function, a bleaching filter applied to the data gives the solution of equally spaced samples located at the zero crossings of the derivative of the phase shifted highest order cosine wave.

APPENDIX A
MINIMUM TRACE APPROACH

(1) MATHEMATICAL FRAMEWORK

To find the optimum times by computer techniques we will first convert the problem to a matrix formulation and the signal, whose spectral content is required, will be modeled as follows for Fourier and Fourier-type signals:

$$X_K(t) = \frac{A_0}{2} + \sum_{k=1}^K A_k \cos W_k t + \sum_{k=1}^K B_k \sin W_k t \quad (A-1)$$

For Fourier signals the W_k are harmonically related, otherwise $X_K(t)$ represents a "Fourier-type" signal. It is further assumed that $X_K(t)$ is contaminated with an additive process $N(t)$, giving the observed process

$$z(t) = X_K(t) + N(t) \quad (A-2)$$

Normally, $z(t)$ is available at a finite set of M times values $\{t_i : i=1, 2, \dots, M\}$ where M is greater than or equal to $2K+1$. It becomes convenient to formulate the problem in matrix form, letting

$$z(t_i) = \frac{A_0}{2} + \sum_{k=1}^K A_k \cos W_k t_i + \sum_{k=1}^K B_k \sin W_k t_i + N(t_i) \quad i=1, 2, \dots, M$$

$$\underline{h}_i \cdot \underline{\theta} + N(t_i) \quad (A-3)$$

where

$$\underline{h}_i = \left[\frac{1}{2}, \cos W_1 t_i, \cos W_2 t_i, \dots, \cos W_K t_i, \sin W_1 t_i, \sin W_2 t_i, \dots, \sin W_K t_i \right]$$

$$\text{and } \underline{\theta} = \left[A_0, A_1, A_2, \dots, A_K, B_1, B_2, \dots, B_K \right]^T$$

The entire set of observations can now be written in an M vector as follows:

$$\underline{Z}(M) = H(M) \cdot \underline{\theta} + \underline{N}(M) \quad (A-4)$$

where $\underline{Z}(M)$, $\underline{N}(M)$ and $H(M)$ are given by

$$\underline{Z}(M) = [z(t_M), z(t_{M-1}), \dots, z(t_1)]^T$$

$$N(M) = \begin{bmatrix} N(t_M) & N(t_{M-1}) & \dots & N(t_1) \end{bmatrix}^T$$

$$H(M) = \begin{bmatrix} h_M^T & h_{M-1}^T & \dots & h_1^T \end{bmatrix}^T$$

(2) OPTIMUM ESTIMATOR

The optimal estimate $\hat{\underline{\theta}}$, for the Fourier coefficients, will imply the best linear unbiased estimate of $\underline{\theta}$. Mendel [7] has shown that for random noise $N(M)$ with noise covariance matrix $R(M)$, such an estimate and its corresponding error covariance matrix $P(M)$, are given by:

$$\hat{\underline{\theta}}(M) = \left[H^T(M) R^{-1}(M) H(M) \right]^{-1} H^T(M) R^{-1}(M) \underline{Z}(M) \quad (A-5)$$

$$P(M) = \left[H^T(M) R^{-1}(M) H(M) \right]^{-1} \quad (A-6)$$

Since the diagonal terms of $P(M)$ represent the variance of the spectral estimates, a measure of overall performance, e , for the estimate $\hat{\underline{\theta}}(M)$ can be defined as the trace of the matrix, i.e.,

$$e = \text{trace } [P(M)]$$

Looking at e , we see that it is a function of the times t_i selected. What we wish to do is select the times t_i such that the e is minimized.

(3) COMPUTER FORMULATION

(a) White noise case. The performance e given by the trace of $P(M)$ was found analytically for the two cosine case, and two cosine and two sine wave case for various number of samples. The formulas given in Appendix B allowed a direct evaluation of e that was minimized on a search of a uniform n -dimensional grid.

(b) Colored noise case. When $N(t)$ is colored noise, the covariance matrix $R(M)$ is no longer diagonal. If the noise is wide sense stationary with autocorrelation function $R_{NN}(\tau)$ given by

$$R_{NN}(\tau) = e^{-\alpha|\tau|}$$

then $R(M)$ is given by

$$R(M) = \begin{bmatrix} 1 & e^{-\alpha|t_n - t_{n-1}|} & e^{-\alpha|t_n - t_1|} \\ e^{-\alpha|t_{n-1} - t_n|} & 1 & e^{-\alpha|t_{n-1} - t_1|} \\ \vdots & \vdots & \vdots \\ e^{-\alpha|t_1 - t_n|} & e^{-\alpha|t_1 - t_{n-1}|} & \dots & 1 \end{bmatrix}$$

A computer program was written to evaluate the $P(M)$ given for the special case that $x_k(t)$ contained only a sine and cosine wave of the same frequency. A steepest descent algorithm was used to find the times t_1 , t_2 , and t_3 that would give the minimum trace of $P(M)$.

APPENDIX B
FORMULAE OF THE TRACE FOR PROGRAM IMPLEMENTATION

Two Cosine Waves Case

Let the signal $z(t)$ be given by

$$z(t) = a_1 \cos w_1 t + a_2 \cos w_2 t + N(t)$$

The trace of the error covariance matrix for the estimates of a_1 and a_2 when three, four, and n samples are taken, can be shown to be

(a) three samples $\{t_1, t_2, t_3\}$

$$\text{TRACE} = \frac{\sigma^2}{\Delta} [\cos^2 w_1 t_3 + \cos^2 w_1 t_2 + \cos^2 w_1 t_1 + \cos^2 w_2 t_3 + \cos^2 w_2 t_2 + \cos^2 w_2 t_1]$$

$$\Delta = [(\cos^2 w_1 t_3 + \cos^2 w_1 t_2 + \cos^2 w_1 t_1)(\cos^2 w_2 t_3 + \cos^2 w_2 t_2 + \cos^2 w_2 t_1)] \\ - (\cos w_1 t_3 \cos w_2 t_3 + \cos w_1 t_2 \cos w_2 t_2 + \cos w_1 t_1 \cos w_2 t_1)^2$$

(b) four samples $\{t_1, t_2, t_3, t_4\}$

$$\text{TRACE} = \frac{\sigma^2}{\Delta} [\cos^2 w_1 t_4 + \cos^2 w_1 t_3 + \cos^2 w_1 t_2 + \cos^2 w_1 t_1 \\ + \cos^2 w_2 t_4 + \cos^2 w_2 t_3 + \cos^2 w_2 t_2 + \cos^2 w_2 t_1]$$

$$\Delta = [(\cos^2 w_1 t_4 + \cos^2 w_1 t_3 + \cos^2 w_1 t_2 + \cos^2 w_1 t_1)(\cos^2 w_2 t_4 + \cos^2 w_2 t_3 + \cos^2 w_2 t_2 \\ + \cos^2 w_2 t_1)] \\ - (\cos w_1 t_4 \cos w_2 t_4 + \cos w_1 t_3 \cos w_2 t_3 + \cos w_1 t_2 \cos w_2 t_2 + \cos w_1 t_1 \cos w_2 t_1)^2$$

(c) n samples t_1, t_2, \dots, t_n

$$\text{TRACE} = \frac{\sigma^2}{\Delta} [\cos^2 w_1 t_n + \dots + \cos^2 w_1 t_2 + \cos^2 w_1 t_1 + \cos^2 w_2 t_n + \dots + \cos^2 w_2 t_2 + \cos^2 w_2 t_1]$$

$$\Delta = [(\cos^2 w_1 t_n + \dots + \cos^2 w_1 t_2 + \cos^2 w_1 t_1)(\cos^2 w_2 t_n + \dots + \cos^2 w_2 t_2 + \cos^2 w_2 t_1)] \\ - (\cos w_1 t_n \cos w_2 t_n + \dots + \cos w_1 t_2 \cos w_2 t_2 + \cos w_1 t_1 \cos w_2 t_1)^2$$

Two Cosine and Two Sine Waves

For this case we have

$$z(t) = a_1 \cos w_1 t + a_2 \cos w_2 t + b_1 \sin w_1 t + b_2 \sin w_2 t + N(t)$$

The trace of the error covariance matrix can be found to be the following for the four, five, and n sample case.

(a) four samples $\{t_1, t_2, t_3, t_4\}$

$$\text{TRACE} = \sigma^2 [\text{COF}(1,1) + \text{COF}(2,2) + \text{COF}(3,3) + \text{COF}(4,4)] / \Delta$$

$$\begin{aligned} \Delta = & W_1 X_2 Y_3 Z_4 + X_1 Y_2 Z_3 W_4 + Y_1 Z_2 W_3 X_4 + Z_1 W_2 X_3 Y_4 \\ & - Z_1 Y_2 X_3 W_4 - W_1 Z_2 Y_3 X_4 - X_1 W_2 Z_3 Y_4 - Y_1 X_2 W_3 Z_4 \end{aligned}$$

$$W_1 = A_1^2 + B_1^2 + C_1^2 + D_1^2$$

$$X_1 = A_2 A_1 + B_2 B_1 + C_2 C_1 + D_2 D_1$$

$$W_2 = A_1 A_2 + B_1 B_2 + C_1 C_2 + D_1 D_2$$

$$X_2 = A_2^2 + B_2^2 + C_2^2 + D_2^2$$

$$W_3 = A_1 A_3 + B_1 B_3 + C_1 C_3 + D_1 D_3$$

$$X_3 = A_2 A_3 + B_2 B_3 + C_2 C_3 + D_2 D_3$$

$$W_4 = A_1 A_4 + B_1 B_4 + C_1 C_4 + D_1 D_4$$

$$X_4 = A_2 A_4 + B_2 B_4 + C_2 C_4 + D_2 D_4$$

$$Y_1 = A_3 A_1 + B_3 B_1 + C_3 C_1 + D_3 D_1$$

$$Z_1 = A_4 A_1 + B_4 B_1 + C_4 C_1 + D_4 D_1$$

$$Y_2 = A_3 A_2 + B_3 B_2 + C_3 C_2 + D_3 D_2$$

$$Z_2 = A_4 A_2 + B_4 B_2 + C_4 C_2 + D_4 D_2$$

$$Y_3 = A_3^2 + B_3^2 + C_3^2 + D_3^2$$

$$Z_3 = A_4 A_3 + B_4 B_3 + C_4 C_3 + D_4 D_3$$

$$Y_4 = A_3 A_4 + B_3 B_4 + C_3 C_4 + D_3 D_4$$

$$Z_4 = A_4^2 + B_4^2 + C_4^2 + D_4^2$$

$$\text{COF}(1,1) = X_2 Y_3 Z_4 + Y_2 Z_3 X_4 + Z_2 Y_4 X_3 - Z_2 Y_3 X_4 - Z_3 Y_4 X_2 - Z_4 X_3 Y_2$$

$$\text{COF}(2,2) = W_1 Y_3 Z_4 + Y_1 Z_3 W_4 + Z_1 Y_4 W_3 - Z_1 Y_3 W_4 - Z_3 Y_4 W_1 - Z_4 W_3 Y_1$$

$$\text{COF}(3,3) = W_1 X_2 Z_4 + X_1 Z_2 W_4 + Z_1 X_4 W_2 - Z_1 X_2 W_4 - Z_2 X_4 W_1 - Z_4 W_2 X_1$$

$$\text{COF}(4,4) = W_1 X_2 Y_3 + X_1 Y_2 W_3 + Y_1 X_3 W_2 - Y_1 X_2 W_3 - Y_2 X_3 W_1 - Y_3 W_2 X_1$$

$$A_1 = \cos w_1 t_1$$

$$B_1 = \cos w_1 t_2$$

$$C_1 = \cos w_1 t_3$$

$$D_1 = \cos w_1 t_4$$

$$A_2 = \cos w_2 t_1$$

$$B_2 = \cos w_2 t_2$$

$$C_2 = \cos w_2 t_3$$

$$D_2 = \cos w_2 t_4$$

$$A_3 = \sin w_1 t_1$$

$$B_3 = \sin w_1 t_2$$

$$C_3 = \sin w_1 t_3$$

$$D_3 = \sin w_1 t_4$$

$$A_4 = \sin w_2 t_1$$

$$B_4 = \sin w_2 t_2$$

$$C_4 = \sin w_2 t_3$$

$$D_4 = \sin w_2 t_4$$

(b) five samples $\{t_1, t_2, t_3, t_4, t_5\}$

$$\text{TRACE} = \sigma^2 [\text{COF}(1,1) + \text{COF}(2,2) + \text{COF}(3,3) + \text{COF}(4,4)] / \Delta$$

$$\begin{aligned} \Delta = & W_1 X_2 Y_3 Z_4 + X_1 Y_2 Z_3 W_4 + Y_1 Z_2 W_3 X_4 + Z_1 W_2 X_3 Y_4 \\ & - Z_1 Y_2 X_3 W_4 - W_1 Z_2 Y_3 X_4 - X_1 W_2 Z_3 Y_4 - Y_1 X_2 W_3 Z_4 \end{aligned}$$

$$\text{COF}(1,1) = X_2 Y_3 Z_4 + Y_2 Z_3 X_4 + Z_2 Y_4 X_3 - Z_2 Y_3 X_4 - Z_3 Y_4 X_2 - Z_4 Y_3 X_2$$

$$\text{COF}(2,2) = W_1 Y_3 Z_4 + Y_1 Z_3 W_4 + Z_1 Y_4 W_3 - Z_1 Y_3 W_4 - Z_3 Y_4 W_1 - Z_4 W_3 Y_1$$

$$\text{COF}(3,3) = W_1 X_2 Z_4 + X_1 Z_2 W_4 + Z_1 X_4 W_2 - Z_1 X_2 W_4 - Z_2 X_4 W_1 - Z_4 W_2 X_1$$

$$\text{COF}(4,4) = W_1 X_2 Y_3 + X_1 Y_2 W_3 + Y_1 X_3 W_2 - Y_1 X_2 W_3 - Y_2 X_3 W_1 - Y_3 W_2 X_1$$

$$W_1 = A_1^2 + B_1^2 + C_1^2 + D_1^2 + E_1^2$$

$$x_1 = A_1 A_2 + B_1 B_2 + C_1 C_2 + D_1 D_2 + E_1 E_2$$

$$W_2 = A_1 A_2 + B_1 B_2 + C_1 C_2 + D_1 D_2 + E_1 E_2$$

$$x_2 = A_2^2 + B_2^2 + C_2^2 + D_2^2 + E_2^2$$

$$W_3 = A_1 A_3 + B_1 B_3 + C_1 C_3 + D_1 D_3 + E_1 E_3$$

$$x_3 = A_2 A_3 + B_2 B_3 + C_2 C_3 + D_2 D_3 + E_2 E_3$$

$$W_4 = A_1 A_4 + B_1 B_4 + C_1 C_4 + D_1 D_4 + E_1 E_4$$

$$x_4 = A_2 A_4 + B_2 B_4 + C_2 C_4 + D_2 D_4 + E_2 E_4$$

$$Y_1 = A_1 A_3 + B_1 B_3 + C_1 C_3 + D_1 D_3 + E_1 E_3$$

$$Z_1 = A_1 A_4 + B_1 B_4 + C_1 C_4 + D_1 D_4 + E_1 E_4$$

$$Y_2 = A_2 A_3 + B_2 B_3 + C_2 C_3 + D_2 D_3 + E_2 E_3$$

$$Z_2 = A_2 A_4 + B_2 B_4 + C_2 C_4 + D_2 D_4 + E_2 E_4$$

$$Y_3 = A_3 + B_3 + C_3 + D_3 + E_e$$

$$Z_3 = A_3 A_4 + B_3 B_4 + C_3 C_4 + D_3 D_4 + E_3 E_4$$

$$Y_4 = A_3 A_4 + B_3 B_4 + C_3 C_4 + D_3 D_4 + E_3 E_4$$

$$Z_4 = A_4^2 + B_4^2 + C_4^2 + D_4^2 + E_4^2$$

$$A_1 = \cos w_1 t_1$$

$$B_1 = \cos w_1 t_2$$

$$C_1 = \cos w_1 t_3$$

$$D_1 = \cos w_1 t_4$$

$$E_1 = \cos w_1 t_5$$

$$A_2 = \cos w_2 t_1$$

$$B_2 = \cos w_2 t_2$$

$$C_2 = \cos w_2 t_3$$

$$D_2 = \cos w_2 t_4$$

$$E_2 = \cos w_2 t_5$$

$$A_3 = \sin w_1 t_1$$

$$B_3 = \sin w_1 t_2$$

$$C_3 = \sin w_1 t_3$$

$$D_3 = \sin w_1 t_4$$

$$E_3 = \sin w_1 t_5$$

$$A_4 = \sin w_2 t_1$$

$$B_4 = \sin w_2 t_2$$

$$C_4 = \sin w_2 t_3$$

$$D_4 = \sin w_2 t_4$$

$$E_4 = \sin w_2 t_5$$

(c) n samples $n > 5$

Same formulas as (b) except

$$W_1 = A_1^2 + B_1^2 + \dots + n_1^2$$

$$W_2 = A_1 A_2 + B_1 B_2 + \dots + n_1 n_2$$

$$W_3 = A_1 A_3 + B_1 B_3 + \dots + n_1 n_3$$

$$W_4 = A_1 A_4 + B_1 B_4 + \dots + n_1 n_4$$

$$Y_1 = A_1 A_3 + B_1 B_3 + \dots + n_1 n_3$$

$$Y_2 = A_2 A_3 + B_2 B_3 + \dots + n_2 n_3$$

$$Y_3 = A_3^2 + B_3^2 + \dots + n_3^2$$

$$Y_4 = A_3 A_4 + B_3 B_4 + \dots + n_3 n_4$$

$$X_1 = A_1 A_2 + B_1 B_2 + \dots + n_1 n_2$$

$$X_2 = A_2^2 + B_2^2 + \dots + n_2^2$$

$$X_3 = A_2 A_3 + B_2 B_3 + \dots + n_2 n_3$$

$$X_4 = A_2 A_4 + B_2 B_4 + \dots + n_2 n_4$$

$$Z_1 = A_1 A_4 + B_1 B_4 + \dots + n_1 n_4$$

$$Z_2 = A_2 A_4 + B_2 B_4 + \dots + n_2 n_4$$

$$Z_3 = A_3 A_4 + B_3 B_4 + \dots + n_3 n_4$$

$$Z_4 = A_4^2 + B_4^2 + \dots + n_4^2$$

$$n_1 = \cos w_1 t_n$$

$$n_2 = \cos w_2 t_n$$

$$n_3 = \sin w_1 t_n$$

$$n_4 = \sin w_2 t_n$$

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